BONDS:

Bonds are securities that establish a creditor relationship between the purchasers (creditor) and the issuer (debtor). The issuer receives a certain amount of money in return for the bond, and is obliged to repay the principal at the end of the lifetime of the bond (maturity). Typically, bonds also require coupon or interest payments. Since all these payments are predetermined as part of the contracts, bonds are also called fixed income securities.

A straight bond is one where the purchaser pays a fixed amount of money to buy Bond. At regular periods, he receives an interest payment, called the Coupon payment. The final interest payment along with the principal are paid at the certain date of maturity. Bonds usually pay a standard coupon amount, \( C \), at regular intervals and this represents the interest on the bond. At the maturity of bond, the final payment is made along with the principal amount which is also called Face Value.

A corporation’s long-term debt is usually involves interest only loans. If, for example, a firm wants to borrow $1,000 for 30 years and the actual interest rate on loans with similar risk characteristics is 12 \( \% \), then the firm will pay a total of $120 as interest each year for 30 years and will repay the actual amount of loan i.e. $1,000 after 30 years.

The security that guarantees these payments is called Bond. A bond may involve more than one interest payment during a year.

In the above example, interest payments could be as follows:

- **If the payments are annually, only one payment is paid of $120 each year.**

- **If the payments are semi-annually, two payments will be paid of $60 per year.**

- **If the payments are quarterly, four payments will be paid of $30 per year.**
A bond is characterized by the following items:

**FACE VALUE (FV):** The amount of principal to be repaid at the bond’s maturity date.

**COUPON (Ct):** The amount of interest to be paid each year.

**MATURITY (T):** The number of years until the face value is repaid.

**NUMBER OF PAYMENTS (t):** Number of interest payments in a year.

**ZERO COUPON BONDS:**

A zero-coupon bond is a bond that does not make coupon payments, in other words, it’s a bond with a 0% coupon rate, and it just pays the principal payment or face value at the time of maturity.

The present value of zero-coupon bonds can be computed as,

$$PV = \frac{F_v}{(1 + y)^T}$$
Where,

\[ FV = \text{Future value} / \text{Face value of the bond at the time of maturity.} \]

\[ PV = \text{Present value of the bond.} \]

\[ y = \text{the discounting factor} / \text{market rate} \]

\[ T = \text{the number of periods to final maturity} \]

For instance, the price of a zero-coupon bond with $1000 face value and 22 years to maturity when the return on similar bonds is 6% is,

\[
PV = \frac{1000}{(1 + 0.06)^{22}} = $277.51
\]

Conversely, we can compute the future value of the this bond as,

\[ FV = PV \times (1 + y)^T \]

In the above case, the compounding period was annually, but if there is a condition of compounding semi-annually so the interest rate should be stated along with the method used for compounding semi-annually. The equation of present value will become as,

\[
PV = \frac{FV}{(1 + \frac{y}{2})^{2 \times T}}
\]

Now, we may generalize the above equation for \((m)\) periods compounding in a year as,

\[
PV = \frac{FV}{(1 + \frac{y}{m})^{m \times T}}
\]

Similarly, if the interest rate is compounding continuously than the equation of present value will become as,

\[ PV = FV \times \exp(-y \times T) \]

Zero-coupon bond always sell at par
➢ If the present value of bond is equal to the face value or future value, that is, \( PV = FV \), the bond will be sell at par.

➢ If the present value of bond is greater than the face value or future value, that is, \( PV > FV \), the bond will be sell at premium.

➢ If the present value of bond is less than the face value or future value, that is, \( PV < FV \), the bond will be sell at discount.

**INTEREST RATE SENSITIVITY:**

**ZERO COUPON BONDS:**

Now, we will check the sensitivity of interest rate of zero coupon bonds that what effect it takes on bond price by changing the maturity of these zero-coupon bonds. *For instance*, we have three kinds of zero coupon bonds with maturity of **1-year**, **2-years** and **10 years**, all with the face value of **$1000** and the yield rate **10%** compounded annually.

<table>
<thead>
<tr>
<th>Interest Rate</th>
<th>Bond-1, 1-Year</th>
<th>Bond-2, 2-Years</th>
<th>Bond-3, 10-Years</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.0%</td>
<td>$917.43</td>
<td>$841.68</td>
<td>$422.41</td>
</tr>
<tr>
<td>10.0%</td>
<td>$909.09</td>
<td>$826.45</td>
<td>$385.54</td>
</tr>
<tr>
<td>11.0%</td>
<td>$900.90</td>
<td>$811.62</td>
<td>$352.18</td>
</tr>
</tbody>
</table>

- Bond prices move up if interest rates drop, decrease if interest rates rise.
• Having seen the above graph, we conclude that the bond prices are inversely related to interest rates.

• Long term bonds are more sensitive to Interest rate changes than short term bonds.

• The lower the interest rates, the more sensitive the price of our bonds.

VALUING COUPON BONDS:

A coupon bond is a bond that makes coupon payments periodically, in other words, it’s a bond with a $(c)$ percent coupon rate. This bond is with a fixed cash flow pattern. We can write the present value of bond $P$ as the discounted value of future cash flows:

$$P = \sum_{t=1}^{T-1} \frac{C_t}{(1 + y)^t} + \frac{C_T + F}{(1 + y)^T}$$

Where,

- $F$ = Future value / Face value of the bond at the time of maturity.
- $P$ = Present value of the bond.
- $y$ = the discounting factor / market rate.
- $T$ = the number of periods to final maturity.
- $m$ = the number of payments in each year.

A typical cash flow pattern consists of a regular coupon payment plus the repayment of the principal, or face value at the maturity. Where $c$ is the coupon rate and $F$ is the face value at the time of maturity. We have,

$$C_t = cF$$

Which is prior to expiration, and at expiration, we have

$$C_T = cF + F$$
**PREPETUAL BONDS:**

Consols or perpetual bonds are the bonds which makes regular coupon payments but with no redemption date. For a consol, the maturity is infinite and the cash flows are all equal to a fixed percentage of the face value,

\[ C_t = C = cF \]

Now the price of these types of bonds can be simplified as,

\[
P = cF \left[ \frac{1}{(1+y)} + \frac{1}{(1+y)^2} + \frac{1}{(1+y)^3} + \cdots \right] = \frac{cF}{y}
\]

Consider a bond that pays $1000 in 10 years and a 6% annual coupon. Assume that the next coupon payment is in exactly one year. What is the market value if the yield is 8%?

\[
C_t = cF = 0.06 \times 1000 = 60
\]

\[
P = \sum_{t=1}^{9} \frac{60}{(1+0.08)^t} + \frac{1000 + 60}{(1+0.08)^{10}} = 865.8
\]

Now, Consider the same example with infinite maturity, the market value of consol will be,

\[
P = \frac{C_t}{y} = \frac{60}{0.08} = 750
\]

Now, if the coupon of the bond is paid twice in a year so the present value of that bond can be simplified as,

\[
P = \sum_{t=1}^{T-1} \frac{C_t/2}{(1+y/2)^{2t}} + \frac{C_t/2+F}{(1+y/2)^{2T}}
\]

The above equation show that the coupon will be paid twice in a year, similarly the time periods will be double in a year so the market rate would be half for the semi-annual payments.
Similarly, if there is \( m \) number of payments in a year, so the market value of bonds accordingly will be as,

\[
P = \sum_{t=1}^{T-1} \frac{C_t}{m} \left(1 + \frac{y}{m}\right)^{mt} + \frac{C_T + mF}{\left(1 + \frac{y}{m}\right)^{mT}}
\]

**For Example:** Consider a bond that pays $1000 in 10 years and a 6% annual coupon rate. Assume that the next coupon payment will be paid three months later. What is the market value if the yield is 8%?

Now, in this example the coupon is being paid quarterly, so the number of payments in a year will become \( m=4 \) and the coupon payment will become $15 and similarly the yield rate, having been divided by \( m=4 \), will remain \( y=2\% \).

\[
P = \sum_{t=1}^{39} \frac{60}{4} \left(1 + \frac{0.08}{4}\right)^t + \frac{60}{4} + 1000 \left(1 + \frac{0.08}{4}\right)^{40}
\]

\[
P = \sum_{t=1}^{39} \frac{15}{1 + 0.02)^t} + \frac{15 + 1000}{(1 + 0.02)^{40}} = $863.22
\]

**BOND PRICE DERIVATIVES:**

Let us now assume that if there is change in yield rate of a bond from its initial value so what will be the effect of this change on the bond price. We can compute the price of bond on different yield rate but if the change is very small so we will use some other techniques to recomputed the price of bonds.

The iteration technique could be a very useful shortcut. The non-linear relationship can be approximate by a **Taylor Expansion** around its initial value.

\[
P_1 = P_0 + f^{'}(y_0)\Delta y + \frac{1}{2} f^{''}(y_0) (\Delta y)^2
\]

Where,

\[
f^{'}(y_0) = \frac{\partial P}{\partial y} \text{ is the first derivative and}
\]
\( f''(y_0) = \frac{\partial^2 P}{\partial y^2} \) is the second derivative.

The above equation is only up to second derivative which is because of the very small change in yield rate sometimes, if the change is very much small in yield rates, so the second derivative term can also be negligible then only the first derivative term will give almost good estimation of the new bond price.

**DURATION OF BOND:**

Duration is one of the commonly used tools for measuring price volatility of fixed income securities. Modified duration provides a linear approximation of the percentage price change of a bond for a 100 basis points change in yield.

For example, the price change for an option-free bond (such as non-callable bonds) with a modified of ten years will be 10% for a 100 basis points change in yield. Modified duration does a good job of estimating price sensitivity for small changes in interest rates but is inadequate for estimating price sensitivity for large changes in rates.

Now, we will drive the equation for duration by using the equation of the present value of zero-coupon bond.

\[
P = \frac{F}{\varphi + y^{-p}}
\]

\[
=> F = P \varphi + y^{-p}
\]

\[
\frac{\partial p}{\partial y} = F \frac{(-T)}{\varphi + y^{-p+1}} = \frac{P \varphi + y^{-p} \varphi - T^{-}}{\varphi + y^{-p+1}}
\]

\[
\frac{\partial p}{\partial y} = P \cdot \frac{-T}{1 + y} = -D^* \cdot P
\]

\[
f''(y_0) = -D^* \cdot P
\]

Where \(D^*\) is the modified duration which is the discounted value for one period of bond maturity. In case of zero-coupon bond, the Macaulay duration \(D\) is always equal to the bond’s maturity \(T\).
CONVEXITY OF BONDS:

Convexity measures the rate of change of bond’s price sensitivity to changes in interest rates. It is the rate of change in bond’s duration.

There are several important properties relating to convexity. First, there is an inverse relationship between coupon and convexity (*i.e. lower the coupon bonds have higher the convexity*). Second, there is a direct relationship between maturity of bond and convexity (*i.e. convexity increases maturity increases*). Third, there is an inverse relationship between yield and convexity (*i.e. the lower the yield, the higher the convexity*).

Combining modified duration and convexity significantly improves the estimate of the price volatility of a bond as interest rates change.

Now, we will drive the equation for convexity by using the first derivative equation of present value of zero coupon bonds.

\[
\frac{\partial P}{\partial y} = -D \times P
\]

\[
\frac{\partial P}{\partial y} = -y \times \frac{(1+y)^{T+1}}{(1+y)^{T+1}}
\]

\[
\frac{\partial^2 P}{\partial y^2} = -\frac{(-y)(T+1)}{(1+y)^{T+2}}
\]

\[
\frac{\partial^2 P}{\partial y^2} = \frac{T(T+1)}{(1+y)^{T+2}} \times P = C \times P
\]

\[
f''(y_0) = C \times P
\]

Where, \(C\) is the convexity of bond in case of zero coupon bonds.

Now, the value of both derivatives will be putting in the equation, as we mentioned earlier, of new price of bonds which is given as,

\[
P_1 = P_0 + f'(y_0) \Delta y + \frac{1}{2} \times f''(y_0). (\Delta y)^2
\]

\[
P_1 = P_0 - (D \times P). \Delta y + \frac{1}{2} \times (C \times P). (\Delta y)^2
\]
\[ P_t - P_0 = -(D^* \times P) \Delta y + \frac{1}{2} \times (C \times P) \cdot (\Delta y)^2 \]

\[ \Delta P = -(D^* \times P) \Delta y + \frac{1}{2} \times (C \times P) \cdot (\Delta y)^2 \]

Therefore duration measures the first order (linear) effect of change in yield and convexity measures the second order (quadratic) term.

**Example:**

Consider a 10-years zero coupon bond with yield of 6% compounded semiannually and the present value of bond is $55.368. If there is one percent change in yield then find;

1) Modified Duration
2) Convexity and
3) New price of the bond.

**Solution:**

**Modified Duration.**
As we know that;

\[ D^* = \frac{T}{(1 + y)} = \frac{2T}{(1 + \frac{y}{2})} \]

\[ D^* = \frac{2 \times 10}{(1 + 0.03)} = $19.42 \]

**Convexity,**

\[ C = \frac{T(T + 1)}{(1 + y)^2} = \frac{2T(2T + 1)}{(1 + \frac{y}{2})^2} \]

\[ C = \frac{20(20 + 1)}{(1 + .03)^2} = $395.89 \]
Price of bond will be,

\[ P_1 = P_0 - (D \times P) \Delta y + \frac{1}{2} \times (C \times P) \times (\Delta y)^2 \]

\[ P_1 = 55.368 \left( \frac{19.42}{2} \times 55.368 \right)(0.01) + \frac{1}{2} \left( \frac{395.89}{4} \times 55.368 \right)(0.01)^2 \]

\[ P_1 = \$50.265 \]

**EFFECT OF CONVEXITY:**

As the above figure shows, when the yield rises, the price drops but less than predicted by the tangent. Conversely, if the yield falls, the price increases faster than the duration model. In other words, the quadratic term is always beneficial.

**EFFECTIVE DURATION AND CONVEXITY:**

The bond’s modified duration and convexity can also be computed directly from numerical derivatives. Duration and convexity can not be computed directly for some bonds, such as mortgage-backed securities, because their cash flows are uncertain. Instead, the portfolio manager has access to pricing models that can be used to re-price the securities under various yield environments.
We choose a change in yield, $\Delta y$, and re-price the bond under an up-move scenario,

$$P_+ = P(y_0 + \Delta y)$$

And down-move scenario,

$$P_- = P(y_0 - \Delta y)$$

Effective duration is measured the numerical derivative, Using

$$D^* = -\left( \frac{1}{P} \right) \times \left( \frac{\partial P}{\partial y} \right),$$

It is estimated as,

$$D^E = \frac{P_+ - P_-}{2P_0 \Delta y} = \frac{P(y_0 - \Delta y) - P(y_0 + \Delta y)}{2P_0 \Delta y}$$

Similarly, the Effective Convexity can also be estimated as,

$$C^E = \frac{[D_- - D_+]}{\Delta y}$$

$$C^E = \left[ \frac{P(y_0 - \Delta y) - P_0}{P_0 \Delta y} - \frac{P_0 - P(y_0 + \Delta y)}{P_0 \Delta y} \right] \Delta y$$

**EXAMPLE:**

Consider a 30-years zero-coupon bond with a yield of 6% initially; the initial price of the bond is $17.411. There is one percent change in bond yield, find the effective duration and convexity of the bond.

**Solution:**

$P$. Shows that if the yield rate decreases by one percent, it will be the present value of bond at $y=5\%$. 
$P_+ = \frac{100}{(1.07)^{30}} = $13.14

$P_-$ Shows that if the yield rate increases by one percent, it will be the present value of bond a $y=7\%$.

Now, we can easily find the effective duration of these bonds by substituting these values in the above mentioned formula.

\[
D^E = \frac{P_--P_+}{2P_0\Delta y} = \frac{14.23-13.14}{2 \times 17.411 \times 0.01} = 28.72
\]

Similarly, we can find the effective convexity,

\[
C^E = \left[ \frac{P_--P_0}{P_0\Delta y} - \frac{P_0-P_+}{P_0\Delta y} \right] / \Delta y = \left[ \frac{23.14-17.411}{17.411 \times 0.01} - \frac{17.411-13.14}{17.411 \times 0.01} \right] / 0.01
\]

\[
C^E = 837.40
\]

**COUPON CURVE DURATION:**

If there is change in coupon rate instead of yield rate of a bond so there will be some other technique used in order to evaluate the duration of a bond by considering bond with same maturity but different coupons. If interest rate decreases by 100 basis points, the market price of a 6% 30-year bond should go up, close to that of a 7% 30-year bond.

Thus, we replace a drop in yield of $\Delta y$ by an increase in coupon $\Delta c$ and use the effective duration method to find the Coupon Curve Duration.
This approach is very much useful for those securities that are difficult to price under various yield scenario. Instead, it only requires the market prices of securities with different coupons.

**Example:**

Consider a 10-year bond that pays a 7% coupon semi-annually. In a 7% yield environment, the bond is selling at par. The prices of 6% and 8% coupon bonds are $92.89 and $107.11, respectively. What will be the coupon curve duration of this bond?

**Solution:**

As we know that,

\[
D_{cc} = \frac{P_+ - P_-}{2P_0 \Delta c}
\]

\[
D_{cc} = \frac{P(0.07, c + \Delta c) - P(0.07, c - \Delta c)}{2P_0 \Delta c}
\]

\[
D_{cc} = \frac{107.11 - 92.89}{2 \times 100 \times 0.01} = 7.11
\]

**DURATION AND CONVEXITY:**

**COUPON BONDS:**

Previously, we accumulated the formulas for zero-coupon bonds in which the coupon was not paid, but it is slightly difficult to calculate the duration and convexity for the coupon bond. Because, previously, in case of zero-coupon bond, the duration was equal to its maturity, but in this case, the duration must be lesser than the maturity of these bonds.

As we derived the equation for duration and convexity for zero-coupon bonds, similarly we will derive formulas for the duration and convexity of coupon bonds.

As we know that the present value of coupon bonds is,
\[ P = \sum_{t=1}^{T} \frac{C_t}{(1 + y)^t} \]

\[ \frac{\partial P}{\partial y} = \sum_{t=1}^{T} \frac{-tC_t}{(1 + y)^{t+1}} \]

\[ \frac{\partial p}{\partial y} = \frac{-1}{1 + y} \left( \sum_{t=1}^{T} \frac{tC_t}{(1 + y)^t} \right) \times P \times P \]

\[ \frac{\partial p}{\partial y} = \frac{-D}{1 + y} \times P \]

\[ \text{where, } D = \sum_{t=1}^{T} \frac{t \times C_t}{(1 + y)^t} \]

\[ \frac{\partial p}{\partial y} = -D \times P \]

**Example:**

Consider an 8% coupon bond with maturity of 10 years with the face value is $1000 and the yield rate of the bond is 6% compounded annually. Find the duration of this bond.

<table>
<thead>
<tr>
<th>Coupon Rate</th>
<th>8.00%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Face Value</td>
<td>$1,000</td>
</tr>
<tr>
<td>Yield</td>
<td>6.0000%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Coupon Number</th>
<th>Cash Flow</th>
<th>Present Value</th>
<th>Proportion of Year</th>
<th>Years</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$80.00</td>
<td>75.47</td>
<td>6.58%</td>
<td>1.0</td>
<td>0.07</td>
</tr>
<tr>
<td>2</td>
<td>80</td>
<td>71.20</td>
<td>6.21%</td>
<td>2.0</td>
<td>0.12</td>
</tr>
<tr>
<td>3</td>
<td>80</td>
<td>67.17</td>
<td>5.86%</td>
<td>3.0</td>
<td>0.18</td>
</tr>
<tr>
<td>4</td>
<td>80</td>
<td>63.37</td>
<td>5.52%</td>
<td>4.0</td>
<td>0.22</td>
</tr>
<tr>
<td>5</td>
<td>80</td>
<td>59.78</td>
<td>5.21%</td>
<td>5.0</td>
<td>0.26</td>
</tr>
<tr>
<td>6</td>
<td>80</td>
<td>56.40</td>
<td>4.92%</td>
<td>6.0</td>
<td>0.29</td>
</tr>
<tr>
<td>7</td>
<td>80</td>
<td>53.20</td>
<td>4.64%</td>
<td>7.0</td>
<td>0.32</td>
</tr>
</tbody>
</table>
Another Example:

Consider, another 8% coupon bond with the maturity of 5-years compounded semiannually with the face value of $1000 and the yield rate of the bond is 6%. Find the duration of the bond.

<table>
<thead>
<tr>
<th>Coupon Rate</th>
<th>8.00%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Face Value</td>
<td>$1,000</td>
</tr>
<tr>
<td>Yield</td>
<td>6.0000%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Coupon Number</th>
<th>Cash Flow</th>
<th>Present Value</th>
<th>Propn. of Years</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$40.00</td>
<td>38.83</td>
<td>3.58%</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td>37.70</td>
<td>3.47%</td>
<td>1.0</td>
</tr>
<tr>
<td>3</td>
<td>40</td>
<td>36.61</td>
<td>3.37%</td>
<td>1.5</td>
</tr>
<tr>
<td>4</td>
<td>40</td>
<td>35.54</td>
<td>3.27%</td>
<td>2.0</td>
</tr>
<tr>
<td>5</td>
<td>40</td>
<td>34.50</td>
<td>3.18%</td>
<td>2.5</td>
</tr>
<tr>
<td>6</td>
<td>40</td>
<td>33.50</td>
<td>3.09%</td>
<td>3.0</td>
</tr>
<tr>
<td>7</td>
<td>40</td>
<td>32.52</td>
<td>3.00%</td>
<td>3.5</td>
</tr>
<tr>
<td>8</td>
<td>40</td>
<td>31.58</td>
<td>2.91%</td>
<td>4.0</td>
</tr>
<tr>
<td>9</td>
<td>40</td>
<td>30.66</td>
<td>2.82%</td>
<td>4.5</td>
</tr>
<tr>
<td>10</td>
<td>1040</td>
<td>773.86</td>
<td>71.30%</td>
<td>5.0</td>
</tr>
</tbody>
</table>

Bond Value: 1147.202 Duration: 7.45 Modified Duration: 7.23

Bond Value: 1085.302 Duration: 4.25 Modified Duration: 4.13
**SENSITIVITY OF DURATION:**

Now, we will take different coupon payments with constant yield and face value of the bond, we will find out the duration with different maturity and watch the behavior of duration as the coupon of the bond changes.

Consider two bonds, one has maturity of 5 year and the other has 10-year maturity with 6% yield rate. The duration of the bond with different coupon rates will be as follow,

<table>
<thead>
<tr>
<th>Coupon</th>
<th>Duration 5-year</th>
<th>Duration 10-year</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>4.76</td>
<td>9.12</td>
</tr>
<tr>
<td>4</td>
<td>4.56</td>
<td>8.51</td>
</tr>
<tr>
<td>6</td>
<td>4.39</td>
<td>8.07</td>
</tr>
<tr>
<td>8</td>
<td>4.25</td>
<td>7.73</td>
</tr>
<tr>
<td>10</td>
<td>4.14</td>
<td>7.47</td>
</tr>
<tr>
<td>12</td>
<td>4.03</td>
<td>7.25</td>
</tr>
<tr>
<td>14</td>
<td>3.94</td>
<td>7.08</td>
</tr>
<tr>
<td>16</td>
<td>3.87</td>
<td>6.93</td>
</tr>
<tr>
<td>18</td>
<td>3.8</td>
<td>6.8</td>
</tr>
<tr>
<td>20</td>
<td>3.73</td>
<td>6.7</td>
</tr>
</tbody>
</table>

**Coupon and Duration Graph**

![Coupon and Duration Graph](image-url)
As expected, the bond with the higher coupon rate has a shorter duration; this above example and graph illustrate the two important properties of duration.

- **First, the duration of bond is less than its time to maturity (except zero-coupon bonds).**

- **Second, the duration of the bond decreases the greater the coupon rate.**

This is because more weight (present value weight) is being given to the coupon payments. A third property is that, as market interest rate increases, the duration of the bond decreases. This is a direct result of discounting. Discounting at a higher rate means lower weight on payments in the far future.

**Practical use of Modified Duration:**

<table>
<thead>
<tr>
<th>Bond Price</th>
<th>6% Bond</th>
<th>10% Bond</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modified Duration</td>
<td>4.19</td>
<td>7.08</td>
</tr>
</tbody>
</table>

Now assume that the yield increases to **6.5%**. Let’s examine the change in the value of each bond using two different approaches.

**METHOD-1**

*Using Modified Duration:*

Using this method the percentage change in the value of each bond is the product of the modified duration and the change in the yield,

<table>
<thead>
<tr>
<th>6% Bond</th>
<th>10% Bond</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modified Duration</td>
<td>4.19</td>
</tr>
<tr>
<td>Change in yield</td>
<td>0.50%</td>
</tr>
<tr>
<td>% change in price</td>
<td>2.10%</td>
</tr>
<tr>
<td>New bond price</td>
<td>899.62</td>
</tr>
</tbody>
</table>

Note that the price of the 6 percent bond has gone down proportionally more than the 10 percent bond. This is because the 6% bond has higher duration.
METHOD-2:
*Compute directly the new bond prices:*

<table>
<thead>
<tr>
<th></th>
<th>6% Bond</th>
<th>10% Bond</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>New bond price</strong></td>
<td>97.89</td>
<td>114.74</td>
</tr>
</tbody>
</table>

As you can see, Method-1 provides an accurate approximation to the change in bond’s prices.

**CONVEXITY OF COUPON BONDS:**

Convexity usually measures how much a bond price-yield curve deviates from a straight line. Convexity is the second derivative of the bond price with respect to yield rate divided by the bond price,

\[
Convexity = \frac{1}{P} \frac{\partial^2 p}{\partial y^2}
\]

It is sometimes very important calculation to make for a bond because it allows us to improve the duration approximation for bond price changes.

Now, for the derivation of the convexity, we will have to reconsider the equation of first derivative of present value.

\[
\frac{\partial p}{\partial y} = \sum_{t=1}^{T} \frac{-tC_t}{(1 + y)^{t+1}}
\]

Now, the above term will be differentiated again in order to find the quadratic term of convexity.

\[
\frac{\partial^2 p}{\partial y^2} = \sum_{t=1}^{T} \frac{(t)(t + 1)C_t}{(1 + y)^{t+2}}
\]
\[ \frac{\partial^2 P}{\partial y^2} = \frac{\sum_{t=1}^{T} (t)(t+1)C_t}{(1 + y)^{t+2}} \times P \]

\[ \frac{\partial^2 P}{\partial y^2} = C \times P \]

\[ \rightarrow C = \frac{1}{P} \frac{\partial^2 P}{\partial y^2} \]

**Example:**

Consider a bond for 5 years maturity from now, which pays 6 percent coupon semiannually. The yield rate of the bond is 8% therefore the bond is selling at discount. Find the duration and convexity of $1000 bond and also examine the new price of the bond if the yield rate decreases by one percent

\( a) \) Price excluding convexity

\( b) \) Price Including convexity

Which price of bond will be more precise?

**Solution:**

<table>
<thead>
<tr>
<th>Coupon Rate</th>
<th>6.00%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Face Value</td>
<td>$1,000</td>
</tr>
<tr>
<td>Yield</td>
<td>8.0000%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Coupon Number</th>
<th>Cash Flow</th>
<th>Present Value</th>
<th>Proportion of PV</th>
<th>Years</th>
<th>Duration</th>
<th>Convexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$30.00</td>
<td>28.85</td>
<td>3.14%</td>
<td>0.5</td>
<td>0.02</td>
<td>53.34</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>27.74</td>
<td>3.02%</td>
<td>1.0</td>
<td>0.03</td>
<td>153.86</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>26.67</td>
<td>2.90%</td>
<td>1.5</td>
<td>0.04</td>
<td>295.89</td>
</tr>
<tr>
<td>4</td>
<td>30</td>
<td>25.64</td>
<td>2.79%</td>
<td>2.0</td>
<td>0.06</td>
<td>474.19</td>
</tr>
<tr>
<td>5</td>
<td>30</td>
<td>24.66</td>
<td>2.68%</td>
<td>2.5</td>
<td>0.07</td>
<td>683.93</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
<td>23.71</td>
<td>2.58%</td>
<td>3.0</td>
<td>0.08</td>
<td>920.67</td>
</tr>
</tbody>
</table>
As we already know the formula for the new price of a bond is,

\[ P_1 = P_0 - (D \times P) \Delta y + \frac{1}{2} \times (C \times P) \times (\Delta y)^2 \]

a) Price Excluding Convexity.

\[ P_1 = P_0 - (D \times P) \Delta y \]

\[ P_1 = 918.891 - (4.19 \times 918.891) \times 0.01 \]

\[ P_1 = $880.40 \]

b) Price including convexity.

\[ P_1 = 880.40 + (0.5 \times 21.15 \times 918.891) \times 0.01^2 \]

\[ P_1 = $881.36 \]

In the above example, when the market rate changes so how does the convexity effect the new bond’s price. In part-a, the price of bond includes only the duration term with the effect of changing yield which is effective but a little under estimation without convexity but when we included the convexity term in the approximation of bond price, the price little increases and near to the actual price of bond.

**INTERPRETATION OF CONVEXITY:**

The behavior of convexity can be analyzed by comparing the bond with zero-coupon bond to a 6 percent coupon bond with identical maturity.
The Zero-coupon bond always has greater convexity than a coupon bond, because there is only one cash flow at maturity. Its convexity is roughly the square of maturity, for example about 100 for a 10-year zero. In contrast, the 10-year coupon bond has a convexity of about 70 only. The convexity will always be positive for a bond whose cash flow is certain while the convexity can be negative for bonds that have uncertain cash flows, such as mortgage-backed securities (MBSs) or callable bonds.
FINAL ANALYSIS:

Now, consider a real scenario, in which we have the data of Pakistan Investment Bonds (PIB), with different maturity and yield rate.

Assume that all the bonds are selling at par and their face value is Rs. 1000.

<table>
<thead>
<tr>
<th>PIB</th>
<th>Years to Maturity</th>
<th>Yield rate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3 yrs</td>
<td>9.37</td>
</tr>
<tr>
<td></td>
<td>5 yrs</td>
<td>9.55</td>
</tr>
<tr>
<td></td>
<td>6 yrs</td>
<td>9.65</td>
</tr>
<tr>
<td></td>
<td>7 yrs</td>
<td>9.73</td>
</tr>
<tr>
<td></td>
<td>8 yrs</td>
<td>9.84</td>
</tr>
<tr>
<td></td>
<td>9 yrs</td>
<td>9.9</td>
</tr>
<tr>
<td></td>
<td>15 yrs</td>
<td>11</td>
</tr>
</tbody>
</table>

Their present value, duration and convexity are respectively calculated below by using the above methodologies.

<table>
<thead>
<tr>
<th>PIB</th>
<th>Years to Maturity</th>
<th>Yield rate</th>
<th>PV</th>
<th>duration</th>
<th>Modified duration</th>
<th>convexity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3 yrs</td>
<td>9.37</td>
<td>1000</td>
<td>2.75</td>
<td>2.51</td>
<td>8.92</td>
</tr>
<tr>
<td></td>
<td>5 yrs</td>
<td>9.55</td>
<td>1000</td>
<td>4.20</td>
<td>4.01</td>
<td>19.73</td>
</tr>
<tr>
<td></td>
<td>6 yrs</td>
<td>9.65</td>
<td>1000</td>
<td>4.82</td>
<td>4.60</td>
<td>25.95</td>
</tr>
<tr>
<td></td>
<td>7 yrs</td>
<td>9.73</td>
<td>1000</td>
<td>5.39</td>
<td>5.14</td>
<td>32.51</td>
</tr>
<tr>
<td></td>
<td>8 yrs</td>
<td>9.84</td>
<td>1000</td>
<td>5.89</td>
<td>5.62</td>
<td>8.26</td>
</tr>
<tr>
<td></td>
<td>9 yrs</td>
<td>9.90</td>
<td>1000</td>
<td>6.35</td>
<td>6.05</td>
<td>46.21</td>
</tr>
<tr>
<td></td>
<td>15 yrs</td>
<td>11.00</td>
<td>1000</td>
<td>7.98</td>
<td>7.57</td>
<td>79.39</td>
</tr>
</tbody>
</table>

By utilizing the above results, the investors may have no hesitation to choose the bonds in investing their assets according to their benchmark. These computations mean a lot for the purpose of investments. In the real scenario, the benchmarked bond’s duration and convexity do not meet the actual bond’s duration and convexity so the solution of this problem can be drawn by investing the monetary in those bonds which are nearby the preset goal. The other purpose could be investing in some short duration bonds is that if any liability occurs so the investors must have enough liquidity to meet them.